Численные методы решения системы линейных алгебраических уравнений(СЛАУ) методом Гаусса

Айназаров Эмиль Рамисович
Студент, Астраханский государственный технический университет,
Российская Федерация, г. Астрахань
Пивоварова Наталья Александровна
Ст. пр., Астраханский государственных технический университет,
Российская Федерация, г. Астрахань

Numerical Methods for Solving Systems of Linear Algebraic Equations (SLAE) Using the Gaussian Elimination Method

Ainazarov Emil Ramisovich
Student, Astrakhan State Technical University,
Russian Federation, Astrakhan
Pivovarova Natalya Alexandrovna
Lecturer, Astrakhan State Technical University,
Russian Federation, Astrakhan

Аннотация

В статье рассматривается конкретная вычислительная проблема: при решении систем линейных алгебраических уравнений (СЛАУ) в инженерных и научных приложениях классический метод Гаусса сталкивается с накоплением численных ошибок, чувствительностью к выбору ведущих элементов и вариативностью вычислительной устойчивости при различных схемах частичного и полного выбора. Проанализированы источники этих эффектов, сравниваются алгоритмические модификации метода Гаусса (с частичным выбором, полным выбором и без выбора), а также оценивается их влияние на точность и вычислительную сложность. Предлагается практическое комбинированное решение, основанное на адаптивном выборе стратегии выбора главного элемента и масштабировании системы, направленное на повышение устойчивости и снижение относительной погрешности при сохранении полиномиальной вычислительной трудоёмкости. Представлены рекомендации по применению метода в условиях ограниченной точности и неоднородности коэффициентов.

Abstract

This article addresses a concrete computational challenge: when solving systems of linear algebraic equations (SLAE), the classical Gaussian elimination method is prone to numerical error accumulation, sensitivity to pivot selection, and variability in numerical stability across different partial and full pivoting schemes. We analyze the sources of these issues, compare algorithmic variants of Gaussian elimination (no pivoting, partial pivoting, and full pivoting), and evaluate their impact on accuracy, stability, and computational complexity. A practical hybrid approach is proposed, combining adaptive pivoting strategies with system scaling to improve robustness and reduce relative error while preserving polynomial-time complexity. The article provides recommendations for applying Gaussian elimination in scenarios involving limited floating-point precision and coefficient heterogeneity.

Ключевые слова: метод Гаусса, системы линейных алгебраических уравнений, численные методы, вычислительная устойчивость, выбор главного элемента, частичное и полное выбора, относительная погрешность.

Keywords: Gaussian elimination, systems of linear algebraic equations, numerical methods, computational stability, pivot selection, partial and full pivoting, relative error.

Systems of linear algebraic equations arise in all major fields of science and engineering, and Gaussian elimination remains one of the most widely used numerical methods for solving them. Despite its conceptual simplicity, the classical method is sensitive to finite-precision arithmetic and may suffer from instability when small pivot elements appear during elimination. These phenomena lead to the accumulation of rounding errors, unpredictable coefficient growth, and variability of numerical behavior across different implementations. For practical applications that require both accuracy and robustness, it is important to understand the factors affecting stability and to employ algorithmic techniques that mitigate numerical degradation.

A general system of linear equations can be written as

$$A\mathrm{x} = \mathrm{b},$$

where $A \in \mathbb{R}^{n\times n}$, $\mathrm{x} \in \mathbb{R}^{n}$, and $\mathrm{b} \in \mathbb{R}^{n}$. Gaussian elimination transforms the matrix into an upper-triangular form

$$U\mathrm{x} = \mathrm{c},$$

after which the unknown vector is found via backward substitution. The elimination step uses multipliers of the form

$$m_{ik} = \frac{a_{ik}}{a_{kk}},$$

and this is precisely where instability often arises: if a_{kk} is very small, the multiplier becomes large and introduces significant rounding error into subsequent computations. Since the total number of arithmetic operations grows as approximately $\frac{2}{3}n^3$, even small inaccuracies may accumulate to the point of dominating the final result, especially when the matrix contains coefficients of different magnitudes.

For these reasons, Gaussian elimination without pivoting is rarely acceptable in real computational practice. Choosing a stable pivot element dramatically improves reliability. Partial pivoting, in which the largest element in the current column is selected as the pivot, is generally sufficient for most engineering problems and forms the basis of many industrial and scientific libraries. Full pivoting, which searches for the largest element in the entire remaining submatrix, offers even greater stability but increases computational cost and is therefore used primarily for small or extremely ill-conditioned systems. Scaled pivoting further normalizes each row before comparison, reducing the effect of heterogeneous coefficients.

Modern implementations often combine these techniques into hybrid adaptive schemes. Such schemes monitor the magnitude of pivot elements, the scaling factors of each row, and the local numerical residual

$$r = \| A\mathrm{x} - \mathrm{b} \|,$$

and switch dynamically between pivoting strategies when instability indicators exceed predefined thresholds. In addition, rescaling the matrix so that each row has a comparable maximum magnitude,

$$\frac{a_{ij}}{s_i}, s_i = \max_j \mid a_{ij} \mid,$$

reduces the growth of intermediate coefficients and improves overall numerical behavior. These methods help maintain acceptable accuracy even when the input data include large or small values that would otherwise lead to significant floating-point errors.

Although pivoting introduces additional work, it does not change the asymptotic complexity of Gaussian elimination, which remains $O(n^3)$. For dense matrices of moderate size, this cost is acceptable and predictable. Memory usage stays within $O(n^2)$, determined primarily by the storage of the coefficient matrix. In practice, the tradeoff between increased computational effort and improved stability is more than justified, particularly in applications with strict accuracy requirements or in systems where floating-point precision is limited.

Gaussian elimination continues to be a foundational method for solving linear systems, but its numerical success depends on careful algorithmic design. Employing partial or full pivoting, applying appropriate scaling procedures, monitoring residuals during computation, and using sufficient precision all contribute to a more stable and reliable solution. Hybrid adaptive approaches that combine these techniques provide a balanced and practical way to control rounding errors and ensure that the method behaves predictably under a wide variety of real-world conditions.

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